

A characterization of quaternionic projective space by the conformal-Killing equation

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Abstract: We prove that a compact quaternionic-Kähler manifold of dimension $4n \geq 8$ admitting a conformal-Killing 2-form which is not Killing, is isomorphic to the quaternionic projective space, with its standard quaternionic-Kähler structure.

1 Introduction

The existence of a non-constant smooth function on a Riemannian manifold, which satisfies a well-chosen differential equation can, under certain conditions, determine the Riemannian manifold. Such characterizations in terms of solutions of differential equations exist for the sphere, the complex projective space and the quaternionic projective space (with their standard metrics), see [7], [17], [22]. In this note we develop an alternative characterization of the quaternionic projective space using the conformal-Killing equation.

Recall that a p -form u on a Riemannian manifold (M^m, g) is Killing, if its covariant derivative ∇u with respect to the Levi-Civita connection ∇ is totally skew-symmetric, or, equivalently, if $\nabla u = \frac{1}{p+1} du$. Killing forms are natural generalizations of Killing vector fields and were introduced for the first time by K. Yano in [23]. More generally, one can consider conformal-Killing forms, which are forms $\psi \in \Omega^p(M)$ satisfying the conformal-Killing equation:

$$\nabla_Y \psi = \frac{1}{p+1} i_Y d\psi - \frac{1}{m-p+1} Y \wedge \delta\psi, \quad \forall Y \in TM. \quad (1)$$

(Here and everywhere in this note we identify tangent vectors with their dual 1-forms, using the Riemannian metric). Note that a conformal-Killing form is

Killing if it is coclosed. There is an intensive literature on conformal-Killing forms. On Kähler manifolds conformal-Killing forms are closely related to Hamiltonian 2-forms and have been completely classified (in the compact case) in [16]. In particular, conformal-Killing forms exist on Bochner-flat Kähler manifolds and on conformally Einstein Kähler manifolds. In this paper, we study conformal-Killing 2-forms on quaternionic-Kähler manifolds, which are Riemannian manifolds with holonomy group included in $\mathrm{Sp}(n)\mathrm{Sp}(1)$. It is known that on a compact quaternionic-Kähler manifold any Killing p -form (with $p \geq 2$) is parallel [15]. For this reason, we shall be interested in conformal-Killing 2-forms which are not Killing. Our aim is to prove the following result:

Theorem 1. *1. A compact, connected, quaternionic-Kähler manifold (M, g) of dimension $4n \geq 8$ admits a conformal Killing 2-form which is not Killing if and only if it is isomorphic to the quaternionic projective space $\mathbb{H}P^n$, with its standard quaternionic-Kähler structure.*

2. Let $g_{\mathrm{can}}(\nu)$ be the standard metric of $\mathbb{H}P^n$, with reduced scalar curvature $\nu > 0$. The map which associates to a Killing vector field X on $(\mathbb{H}P^n, g_{\mathrm{can}}(\nu))$ the 2-form

$$\psi := -\frac{2}{\nu(4n-1)}(\nabla X)^{S^2H} + \frac{4}{\nu(4n-1)}(\nabla X)^{S^2E} \quad (2)$$

is an isomorphism from the space of Killing vector fields to the space of conformal-Killing 2-forms on $(\mathbb{H}P^n, g_{\mathrm{can}}(\nu))$, with inverse the codifferential: $\delta(\psi) = X$.

The plan of the paper is the following. In Section 2 we recall some basic facts about quaternionic-Kähler manifolds and the conformal-Killing equation. An important feature for us is that the codifferential of a conformal-Killing 2-form on an Einstein manifold (hence also on a quaternionic-Kähler manifold) is a Killing vector field [20]. In Section 3 we determine the general form of conformal-Killing 2-forms on a compact quaternionic-Kähler manifold (M, g) of dimension $4n \geq 8$ (see Proposition 2). We show that there are no conformal-Killing, non-Killing 2-forms on (M, g) , unless the reduced scalar curvature ν is positive, in which case a Killing vector field X on (M, g) is the codifferential of a conformal-Killing 2-form if and only if it belongs to the kernel of the quaternionic-Weyl tensor W of (M, g) . In Section 4 we conclude the proof of Theorem 1, by showing that if (M, g) (still compact, quaternionic-Kähler, with $\nu > 0$ and of dimension $4n \geq 8$) admits a (non-trivial) Killing vector field X in the kernel of W , then it is isometric to the standard quaternionic projective space (see Proposition 9). Our method to

prove this statement is to consider the Hamiltonian function f^X of the natural lift X^Z of X to the twistor space Z of (M, g) and to show that it satisfies a certain differential equation introduced by Obata in [17]. By a result of Obata (see [17], Theorem *C*) this implies that Z (with its standard Kähler-Einstein structure) is isomorphic to the complex projective space, with its Fubini-Study metric and therefore the quaternionic-Kähler manifold (M, g) must be isomorphic to $(\mathbb{H}P^n, g_{\text{can}}(\nu))$.

Similar type of results appear in the literature. In four dimensions, a conformal oriented manifold with degenerate and coclosed self-dual Weyl tensor is necessarily anti-self-dual (see [6], page 454). In the same framework, gradient quaternionic vector fields on quaternionic-Kähler manifolds lie in the kernel of the quaternionic-Weyl tensor; moreover, if the quaternionic-Kähler manifold is compact and non Ricci-flat, then it has no non-zero gradient vector fields, unless it is isomorphic to the quaternionic projective space, with its standard quaternionic-Kähler structure, see [1], [2].

In the last Section of the paper we determine the dimension of the space of conformal-Killing 2-forms defined on a compact, quaternionic-Kähler manifold. This is a consequence of Proposition 2 of Section 3.

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2 Preliminary material

In this paper we shall use the following conventions and notations. All our manifolds will be smooth and connected. The space of smooth sections of a vector bundle (real or complex) $V \rightarrow M$ over a manifold M will be denoted by $\Gamma(V)$. As in Introduction, $\Omega^k(M)$ will denote the space of smooth, real-valued k -forms on M . Finally, all our quaternionic-Kähler manifolds will be of (real) dimension bigger or equal to eight.

Quaternionic-Kähler manifolds. Let (M, g) be a quaternionic-Kähler manifold, i.e. a Riemannian manifold with holonomy group included in $\text{Sp}(n)\text{Sp}(1)$. The endomorphism bundle of TM has a distinguished, parallel (i.e. preserved by the Levi-Civita connection) rank 3-subbundle Q , called

the quaternionic bundle, which is locally generated by a system of three almost complex structures $\{J_1, J_2, J_3\}$ (called an admissible basis of Q) subject to the quaternionic relations

$$J_1^2 = J_2^2 = J_3^2 = -\text{Id}, \quad J_i J_j = -J_j J_i, \quad \forall i \neq j.$$

Like for conformal 4-manifolds, there are two locally defined complex vector bundles H and E over M , of rank 2 and $2n$ respectively, associated to the standard representations of $\text{Sp}(1)$ and $\text{Sp}(n)$ on \mathbb{C}^2 and \mathbb{C}^{2n} . The bundles E and H play the role of the spin bundles in conformal geometry. In particular, the complexification $T_{\mathbb{C}}M$ is isomorphic to the tensor product $E \otimes H$ and the complexification of the bundle of 2-forms has the parallel decomposition

$$\Lambda^2(T_{\mathbb{C}}^*M) = S^2H \oplus S^2E \oplus (S^2H \otimes \Lambda_0^2E), \quad (3)$$

where $\Lambda_0^2E \subset \Lambda^2E$ is the kernel of the natural contraction with the standard symplectic form on E . The bundle S^2H is isomorphic to the complexification of the bundle Q , and S^2E is isomorphic to the complexification of the bundle of Q -hermitian forms, i.e. 2-forms ψ which satisfy

$$\psi(AX, Y) = -\psi(X, AY), \quad \forall A \in Q, \quad \forall X, Y \in TM.$$

For a 2-form $\psi \in \Omega^2(M)$, we denote by ψ^{S^2H} , ψ^{S^2E} and $\psi^{S^2H \otimes \Lambda_0^2E}$ its projections on the three factors of the decomposition (3). Note that, when $\psi = X \wedge Y$ is decomposable,

$$(X \wedge Y)^{S^2H} = \frac{1}{2n} \sum_{i=1}^3 \omega_i(X, Y) \omega_i$$

and

$$(X \wedge Y)^{S^2E} = \frac{1}{4} \left(X \wedge Y + \sum_{i=1}^3 J_i X \wedge J_i Y \right),$$

with respect to any admissible basis $\{J_1, J_2, J_3\}$ of Q , with associated Kähler forms $\omega_i = g(J_i \cdot, \cdot)$.

We now turn to the curvature of the quaternionic-Kähler manifold (M, g) . The metric g is Einstein and its curvature has the expression

$$R^g(X, Y) = -\frac{\nu}{4} \left(X \wedge Y + \sum_{i=1}^3 J_i X \wedge J_i Y + 2 \sum_{i=1}^3 \omega_i(X, Y) \omega_i \right) + W(X, Y) \quad (4)$$

where $\nu := \frac{k}{4n(n+2)}$ is the reduced scalar curvature (k being the usual scalar curvature) and W is the quaternionic-Weyl tensor, which is a symmetric endomorphism of S^2E and is in the kernel of the Ricci contraction. The tensor W plays the role of the Weyl tensor in conformal geometry, in the sense that if $W = 0$, then (M, Q) is locally isomorphic, as a quaternionic manifold, to the quaternionic projective space $\mathbb{H}P^n$.

There is one more piece of information we need to recall, namely the twistor space of (M, g) . The bundle Q has a natural Euclidian metric $\langle \cdot, \cdot \rangle$, for which any admissible basis is orthonormal. The twistor space of (M, g) is the total space Z of the unit sphere bundle of Q , i.e. the set of all complex structures of tangent spaces of M , which, seen as endomorphisms of TM , belong to Q . The fibers $Z_p := \pi^{-1}(p)$ of the twistor projection $\pi : Z \rightarrow M$, called twistor lines, are complex manifolds, with complex structure \mathcal{J} defined by

$$\mathcal{J}(A) := J \circ A, \quad \forall A \in T_J Z_p, \quad \forall J \in Z_p, \quad \forall p \in M. \quad (5)$$

Note that \mathcal{J} is well-defined, since

$$T_J Z_p = \{A \in Q_p : A \circ J + J \circ A = 0\} = J^\perp \subset Q_p,$$

where \perp denotes the orthogonal complement with respect to the metric $\langle \cdot, \cdot \rangle$. The twistor space Z has a standard (integrable) complex structure, also denoted by \mathcal{J} , and, when $\nu > 0$, a Kähler-Einstein metric \bar{g} . In order to define \mathcal{J} and \bar{g} , consider the horizontal bundle $H^\nabla \subset TZ$ associated to the Levi-Civita connection ∇ , acting on the twistor bundle $\pi : Z \rightarrow M$. The complex structure \mathcal{J} preserves H^∇ and the twistor lines. Its restriction to H_J^∇ (for any $J \in Z$) is defined tautologically, using the linear isomorphism $\pi_* : H_J^\nabla \rightarrow T_p M$ (where $p := \pi(J)$) and its restriction to the twistor lines coincides with the standard complex structure of the twistor lines, defined in (5). The metric \bar{g} is defined in the following way: on H^∇ it is the pull-back of g ; the twistor lines are \bar{g} -orthogonal to H^∇ ; when $\nu = 1$ the restriction of \bar{g} to the twistor lines is the standard metric on S^2 of curvature one; equivalently, the restriction of \bar{g} to a twistor line Z_p is induced by the Euclidian metric $\langle \cdot, \cdot \rangle$ of the fiber Q_p of Q over p , by means of the inclusion $Z_p \subset Q_p$. The twistor projection $\pi : (Z, \bar{g}) \rightarrow (M, g)$ becomes a Riemannian submersion with totally geodesic fibers.

The conformal-Killing equation. The conformal-Killing equation on p -forms on a compact Riemannian manifold (M^m, g) can be written in the

alternative form [20]

$$q(R)\psi = \frac{p}{p+1}\delta d\psi + \frac{m-p}{m-p+1}d\delta\psi, \quad (6)$$

where $q(R)$ is a bundle endomorphism of $\Lambda^p(T^*M)$, related to the Laplacian $\Delta = d\delta + \delta d$ by the formula $\Delta = \nabla^*\nabla + q(R)$, where, for any form ψ , $\nabla^*\nabla\psi = -\sum_i \nabla^2(\psi)(E_i, E_i)$, $\{E_i\}$ is an orthonormal local frame of TM and $\nabla^2(\psi)(X, Y) := \nabla_X \nabla_Y \psi - \nabla_{\nabla_X Y} \psi$, for any vector fields X and Y . More explicitly,

$$q(R)(\psi) := \sum_{i,j=1}^m E_j \wedge i_{E_i} R^g(E_i, E_j)(\psi), \quad \forall \psi \in \Lambda^p(T^*M). \quad (7)$$

In (7) R^g is the curvature operator of the Levi-Civita connection acting on the form bundle, so that $R^g(E_i, E_j)$ is an endomorphism of $\Lambda^p(T^*M)$ and $R^g(E_i, E_j)(\psi)$ denotes the action of $R^g(E_i, E_j)$ on the p -form ψ . An important feature of the curvature operator $q(R)$ is that it preserves the parallel subbundles of the form bundle (see, for example, [16]). If (M, g) is a symmetric space, then the operator $q(R)$ is parallel and commutes with the Laplace operator Δ . If, moreover, M is compact, Δ acts on the (finite dimensional, see [20]) vector space of conformal-Killing forms on (M, g) and is diagonalisable on this space. In particular, any conformal-Killing form defined on a compact symmetric space can be written as a linear combination of conformal-Killing forms, which are also eigenforms of the Laplace operator. (It is expected that this observation, together with the estimates found in [21] on the eigenvalues of Laplace operator on a compact quaternionic-Kähler manifold with positive scalar curvature, might be useful to understand higher degree conformal-Killing forms on Wolf spaces; further investigation in this direction is needed).

Suppose now that (M^{4n}, g) is a quaternionic-Kähler manifold, with reduced scalar curvature ν . We shall be interested in the conformal-Killing equation on 2-forms defined on (M, g) . The operator $q(R)$ acts on S^2H and $S^2H \otimes \Lambda_0^2 E$ by scalar multiplication as follows (see [21], Lemma 2.5):

$$q(R)|_{S^2H} = 4\nu \text{Id}, \quad q(R)|_{S^2H \otimes \Lambda_0^2 E} = 2\nu(n+2)\text{Id}. \quad (8)$$

(We remark that the operator $q(R)$ of the conformal-Killing equation (6) differs by a multiplicative factor of 2 from the operator $q(R)$ considered in Lemma 2.5 of [21]). The action of $q(R)$ on 2-forms still preserves S^2E , but it is not in general a scalar action on this space.

The metric g being Einstein, the codifferential of a conformal-Killing 2-form on (M, g) is a Killing vector field (this is an easy consequence of Proposition 5.2 of [20]).

3 Conformal Killing 2-forms on quaternionic-Kähler manifolds

On a quaternionic-Kähler manifold with non-zero scalar curvature, the codifferential δ is a linear isomorphism from the space of sections of S^2H which are solutions of the twistor equation to the space of Killing vector fields (see [19], Lemma 6.5 and [4], Proposition 5.6).

In this Section we will prove an analogous statement for conformal-Killing 2-forms. As already mentioned in the Introduction and in Section 2, the codifferential δ sends conformal-Killing 2-forms on a quaternionic-Kähler manifold (M, g) to Killing vector fields and its kernel is the space of Killing 2-forms. We will now determine the image of δ (considered as a map from conformal-Killing 2-forms), under the additional assumption that M is compact. More precisely, we prove the following result.

Proposition 2. *Let ψ be a conformal-Killing 2-form on a compact quaternionic-Kähler manifold (M, g) of dimension $4n \geq 8$ and reduced scalar curvature ν . If $\nu > 0$ then*

$$\psi = -\frac{2}{\nu(4n-1)}(\nabla X)^{S^2H} + \frac{4}{\nu(4n-1)}(\nabla X)^{S^2E} + u, \quad (9)$$

where $X := \delta(\psi)$ is the codifferential of ψ and $u \in \Omega^2(M)$ is parallel. Moreover, $W(X, \cdot) = 0$, where W denotes the quaternionic-Weyl tensor of (M, g) . If $\nu \leq 0$ then ψ is parallel.

Remark 3. For Killing forms, Proposition 2 reduces to the statement proved by Moroianu and Semmelmann in [21], namely that any Killing 2-form on a compact quaternionic-Kähler manifold is parallel. For this reason, Proposition 2 is relevant when ψ is a conformal-Killing, but not Killing 2-form.

We now prove Proposition 2. The case $\nu \leq 0$ is an easy consequence of the following observations: there are no (non-trivial) Killing vector fields on a compact quaternionic-Kähler manifold (M, g) , with $\nu < 0$ (this is an application of the Weinzenböck formula, see also [6], Theorem 1.84). Similarly, if $\nu = 0$ then any Killing vector field on (M, g) is harmonic, and, if coexact, it is identically zero (M being compact). Due to these facts, any

conformal-Killing 2-form on a compact, quaternionic-Kähler manifold with non-negative scalar curvature is Killing, hence parallel [21]. This proves Proposition 2 when $\nu \leq 0$.

It remains to study the case $\nu > 0$. The treatment of this case is more involved and will be divided into several Lemmas. Consider the setting of Proposition 2, with $\nu > 0$. Since M is compact, ψ satisfies the equation

$$\frac{2}{3}\Delta\psi - q(R)\psi + \frac{4(n-1)}{3(4n-1)}dX = 0. \quad (10)$$

In the following Lemma we show that the $S^2H \otimes \Lambda_0^2 E$ -component of ψ is trivial and that its S^2H -component is a solution of the twistor equation. Recall that the map which associates to a Killing vector field X on (M, g) the 2-form $\frac{2}{3\nu}(\nabla X)^{S^2H}$ (sometimes referred as the Hamiltonian form of X) is an isomorphism (in particular, it is injective) from the space of Killing vector fields to the space of solutions of the twistor equation (see [4], Proposition 5.6). Moreover, the Hamiltonian form of any Killing vector field on (M, g) is an eigenform of the Laplace operator Δ , with eigenvalue $2\nu(n+2)$ (see [3], Theorem 2.7).

Lemma 4. *The conformal Killing 2-form ψ is a section of the direct sum bundle $S^2H \oplus S^2E$ and*

$$\psi^{S^2H} = -\frac{2}{\nu(4n-1)}(\nabla X)^{S^2H}. \quad (11)$$

In particular, $\psi^{S^2H} \neq 0$ unless ψ is a Killing 2-form.

Proof. Projecting the conformal-Killing equation (10) on S^2H and using (8) we get

$$\frac{2}{3}\Delta(\psi^{S^2H}) - 4\nu\psi^{S^2H} + \frac{8(n-1)}{3(4n-1)}(\nabla X)^{S^2H} = 0. \quad (12)$$

Define an operator

$$T : \Gamma(Q) \rightarrow \Gamma(Q), \quad T(u) := \frac{2}{3}\Delta u - 4\nu u.$$

The operator T obviously preserves the eigenbundle $E_{2\nu(n+2)}(Q)$ of $\Delta : \Gamma(Q) \rightarrow \Gamma(Q)$, corresponding to the eigenvalue $2\nu(n+2)$, as well as its orthogonal complement $E_{2\nu(n+2)}(Q)^\perp$, taken with respect to the Euclidian metric of $\Gamma(Q)$ defined by the metric $\langle \cdot, \cdot \rangle$ of Q , followed by integration over M . If we write $\psi^{S^2H} = \psi_1 + \psi_2$, with $\psi_1 \in E_{2\nu(n+2)}(Q)$ and $\psi_2 \in E_{2\nu(n+2)}(Q)^\perp$ then,

clearly, $T(\psi_2) = 0$ (because $(\nabla X)^{S^2H} \in E_{2\nu(n+2)}(Q)$, as mentioned above). On the other hand, the eigenvalues of Δ on $\Gamma(Q)$ are bigger or equal to $2\nu(n+2)$ (see [21], Proposition 4.4). Since $6\nu < 2\nu(n+2)$, 6ν cannot be an eigenvalue of Δ on $\Gamma(Q)$. We deduce that $\psi_2 = 0$ and $\psi^{S^2H} \in E_{2\nu(n+2)}(Q)$. Using (12), we now easily get (11). When ψ is conformal-Killing but not Killing, X is non-trivial and $\psi^{S^2H} \neq 0$ (see the comments above).

In a similar way, we prove that ψ is a section of $S^2H \oplus S^2E$, i.e. that $\psi^{S^2H \otimes \Lambda_0^2 E} = 0$. For this, notice that, since X is Killing, $\nabla X \in \Gamma(S^2H \oplus S^2E)$ (see [12], page 247). Projecting (10) on $S^2H \otimes \Lambda_0^2 E$ and using (8) we get

$$\frac{2}{3}\Delta(\psi^{S^2H \otimes \Lambda_0^2 E}) - 2\nu(n+2)\psi^{S^2H \otimes \Lambda_0^2 E} = 0. \quad (13)$$

The eigenvalues of Δ on $\Gamma(S^2H \otimes \Lambda_0^2 E)$ are greater or equal to $4\nu(n+1)$ (see [21], Proposition 4.4). Using (13), we deduce that $\psi^{S^2H \otimes \Lambda_0^2 E} = 0$ when $n > 2$ (because in this case $4\nu(n+1) > 3\nu(n+2)$). It remains to see what happens when $n = 2$. When $n = 2$ (M, g) is a symmetric space (see [14], Theorem 5.4) and, as mentioned in Section 2, ψ can be written as a sum $a_1\psi_1 + \dots + a_k\psi_k$ where ψ_i are conformal-Killing 2-forms and also eigenforms of Laplacian Δ , with eigenvalues, say, ν_i . Some of the ψ_i 's are Killing 2-forms, hence parallel [15]. Others are conformal-Killing, but not Killing 2-forms, and for them the corresponding eigenvalues ν_i are all equal to 8ν , from (11). In particular, ψ is the sum of a parallel 2-form and an eigenform of Δ with eigenvalue 8ν . Using (13) with $n = 2$ we deduce that $\psi^{S^2H \otimes \Lambda_0^2 E} = 0$. \square

In order to prove Proposition 2, it will be useful to write the conformal-Killing equation on 2-forms in an alternative way. We do this in the next Lemma, whose proof is similar to the proof of Proposition 19 of [5].

Lemma 5. *The conformal-Killing 2-form ψ satisfies*

$$\nabla_Y \psi = \frac{1}{4n-1} \left(X \wedge Y + \sum_{k=1}^3 J_k X \wedge J_k Y - \sum_{k=1}^3 \omega_k(X, Y) \omega_k \right) \quad (14)$$

for any $Y \in TM$. Here $\{J_1, J_2, J_3\}$ is an admissible basis of Q with associated Kähler forms $\omega_1, \omega_2, \omega_3$, and, as before, $X = \delta(\psi)$ is the codifferential of ψ .

Proof. In order to verify (14), we will show that

$$d\psi = -\frac{3}{4n-1} (J_1 X \wedge \omega_1 + J_2 X \wedge \omega_2 + J_3 X \wedge \omega_3). \quad (15)$$

Indeed, once we have (15), we replace it into the conformal-Killing equation (1) and we get (14). To prove (15), we define a 3-form

$$\beta := d\psi + \frac{3}{4n-1} (J_1 X \wedge \omega_1 + J_2 X \wedge \omega_2 + J_3 X \wedge \omega_3)$$

and we show that

$$\beta(Y, JV, U) + \beta(Y, V, JU) = 0, \quad \forall Y, U, V \in TM, \quad (16)$$

for a $J \in Z$, which, without loss of generality, can be taken to be J_1 . (It is easy to check that a 3-form with the symmetries (16) must be zero; this implies (15) and our claim).

To show (16), we evaluate $d\psi(Y, JV, U) + d\psi(Y, V, JU)$ using the conformal-Killing equation, written in the form

$$\frac{1}{3}i_Y d\psi = \frac{1}{4n-1}Y \wedge X + \nabla_Y \psi, \quad \forall Y \in TM. \quad (17)$$

Notice that

$$(Y \wedge X)(JV, U) + (Y \wedge X)(V, JU) = -(\omega \wedge JX)(Y, JV, U) - (\omega \wedge JX)(Y, V, JU), \quad (18)$$

where $\omega := \omega_1$ denotes the Kähler form associated to J . Also, using Lemma 4, we can write

$$\begin{aligned} (\nabla_Y \psi)(JV, U) + (\nabla_Y \psi)(V, JU) &= -\frac{2}{\nu(4n-1)} \nabla_Y (\nabla X)^{S^2H}(JV, U) \\ &\quad - \frac{2}{\nu(4n-1)} \nabla_Y (\nabla X)^{S^2H}(V, JU) \\ &\quad + \nabla_Y (\psi^{S^2E})(JV, U) + \nabla_Y (\psi^{S^2E})(V, JU). \end{aligned}$$

But $\nabla_Y (\psi^{S^2E})$ is J -invariant, hence orthogonal to $JV \wedge U + V \wedge JU$ (which is J -anti-invariant) and therefore

$$\nabla_Y (\psi^{S^2E})(JV, U) + \nabla_Y (\psi^{S^2E})(V, JU) = 0.$$

Moreover, from the Konstant formula and the expression (4) of R^g we deduce that

$$\nabla_Y (\nabla X)^{S^2H} = R^g(Y, X)^{S^2H} = -\frac{\nu}{2} \sum_{i=1}^3 \omega_i(Y, X) \omega_i.$$

Combining the above relations, we obtain

$$\begin{aligned} (\nabla_Y \psi)(JV, U) + (\nabla_Y \psi)(V, JU) &= \frac{2}{4n-1} (\omega_2(V, U) \omega_3(Y, X) - \omega_3(V, U) \omega_2(Y, X)) \\ &= -\frac{1}{4n-1} (\alpha(Y, JV, U) + \alpha(Y, V, JU),) \end{aligned}$$

where $\alpha \in \Omega^3(M)$ is defined by

$$\alpha := J_2 X \wedge \omega_2 + J_3 X \wedge \omega_3.$$

Combining the above equality with (17) and (18), we obtain (16) and our claim. \square

Using Lemma 4, we can write our conformal-Killing 2-form ψ as

$$\psi = -\frac{2}{\nu(4n-1)}(\nabla X)^{S^2H} + \frac{4}{\nu(4n-1)}(\nabla X)^{S^2E} + u,$$

where u is a section of S^2E . Equation (14) written in terms of u becomes

$$\nabla_Y u = -\frac{4}{\nu(4n-1)}W(Y, X) \quad \forall Y \in TM. \quad (19)$$

We conclude the proof of Proposition 2 with the following Lemma.

Lemma 6. *Let (M, g) be a compact quaternionic-Kähler manifold, with positive reduced scalar curvature $\nu > 0$ and quaternionic-Weyl tensor W . Let X be an arbitrary vector field on M . Any section $u \in \Gamma(S^2E)$ which satisfies (19) is parallel. In particular, $W(X, \cdot) = 0$.*

Proof. Consider the exterior derivative du , written in the form

$$(du)(Z_0, Z_1, Z_2) = (\nabla_{Z_0} u)(Z_1, Z_2) - (\nabla_{Z_1} u)(Z_0, Z_2) + (\nabla_{Z_2} u)(Z_0, Z_1), \quad (20)$$

where $Z_0, Z_1, Z_2 \in TM$. Using (19), we get

$$\begin{aligned} (du)(Z_0, Z_1, Z_2) &= \frac{4}{\nu(4n-1)}(-W(Z_0, X, Z_1, Z_2) + W(Z_1, X, Z_0, Z_2)) \\ &\quad - \frac{4}{\nu(4n-1)}W(Z_2, X, Z_0, Z_1). \end{aligned}$$

The symmetries of the curvature tensor W imply that $du = 0$. Similarly, we can write the codifferential of u in the form

$$\delta u = -\sum_{i=1}^{4n} (\nabla_{e_i} u)(e_i, \cdot) = \frac{4}{\nu(4n-1)}W(e_i, X)(e_i) = 0,$$

because W is in the kernel of the Ricci contraction. We have proved that u is harmonic. Recall that the second Betti number $b_2(M)$ of (M, g) is zero, unless (M, g) is isomorphic to the Grassmannian $\text{Gr}_2(\mathbb{C}^{n+2})$ of complex 2-planes in \mathbb{C}^{n+2} , with its standard quaternionic-Kähler metric [14]; moreover, the space of harmonic 2-forms on $\text{Gr}_2(\mathbb{C}^{n+2})$ is one dimensional, generated by the Kähler form, which is a parallel section of S^2E . This proves that u is actually parallel. Our claim follows. \square

4 Killing vector fields and the quaternionic-Weyl tensor

In this Section we conclude the proof of Theorem 1. We do this by proving Proposition 9 stated below, which in turn relies on the following Theorem of Obata (see [17], Theorem C).

Theorem 7. *Let (N^{2n}, J, g) be a complete, connected and simply connected Kähler manifold. Suppose there is a non-constant smooth function f on N which satisfies the Obata's equation*

$$4\nabla^2(df)(Y, U, V) = -2df(Y)g(U, V) - df(U)g(Y, V) - df(V)g(Y, U) \\ + df(JU)\omega(Y, V) + df(JV)\omega(Y, U),$$

for any vector fields $Y, U, V \in \mathcal{X}(N)$, where ∇ is the Levi-Civita connection and ω is the Kähler form. Then (N, J, g) is isometric to $(\mathbb{C}P^n, g_{\text{FS}})$, where g_{FS} is the Fubini-Study metric of constant holomorphic sectional curvature equal to one.

Remark 8. It is easy to verify that the Hamiltonian function of any Killing vector field on $(\mathbb{C}P^n, g_{\text{FS}})$ satisfies the Obata's equation. Conversely, Theorem 7 implies that the existence of a *single* Killing vector field on a complete, connected and simply connected Kähler manifold, whose Hamiltonian function is a solution of the Obata's equation, insures that the Kähler manifold is isometric to $(\mathbb{C}P^n, g_{\text{FS}})$.

Proposition 9 below concerns compact quaternionic-Kähler manifolds with positive scalar curvature. Without loss of generality, we will normalise the quaternionic-Kähler metric to have reduced scalar curvature $\nu = 1$. We shall denote by $g_{\text{can}} := g_{\text{can}}(1)$ the standard quaternionic-Kähler metric of $\mathbb{H}P^n$, normalized in this way. The main result of this Section is the following.

Proposition 9. *Let (M, g) be a compact, quaternionic-Kähler manifold, of dimension $4n \geq 8$ and reduced scalar curvature $\nu = 1$. Suppose there is a non-trivial Killing vector field X on M such that $W(X, \cdot) = 0$, where W is the quaternionic-Weyl tensor. Then $W = 0$ and (M, g) is isometric to $(\mathbb{H}P^n, g_{\text{can}})$.*

Remark 10. The idea of the proof of Proposition 9 is to show that the Hamiltonian function f^X of the natural lift X^Z of X to the Kähler-Einstein twistor space $(Z, \bar{g}, \mathcal{J})$ of (M, g) satisfies the Obata's equation stated above. Theorem 7 implies that $(Z, \bar{g}, \mathcal{J})$ is isomorphic to $(\mathbb{C}P^n, g_{\text{FS}})$ and then (M, g) is isomorphic to $(\mathbb{H}P^n, g_{\text{can}})$. Details are as follows.

Since X is Killing, its natural lift X^Z on $(Z, \bar{g}, \mathcal{J})$ is a Killing, real holomorphic vector field, and its value at a point $J \in Z$ is

$$X_J^Z = \bar{X}_J + [\nabla X, J]. \quad (21)$$

In (21) (and in the following considerations), the bar over a tangent vector on M denotes its horizontal lift to Z , using the Levi-Civita connection ∇ of g , acting on the twistor bundle $\pi : Z \rightarrow M$. The comutator $[\nabla X, J]$ is seen as a tangent vertical vector of Z at J , and can be alternatively written as

$$[\nabla X, J] = -2\mathcal{J}(\tilde{A})_J, \quad (22)$$

where $A := (\nabla X)^{S^2H}$ and \tilde{A} is the induced vertical vector field on Z , defined, at a point $J \in Z$, by

$$\tilde{A}_J := A - \langle A, J \rangle J, \quad \forall J \in Z. \quad (23)$$

Therefore, the vector field X^Z has the form

$$X^Z = \bar{X} - 2\mathcal{J}(\tilde{A}). \quad (24)$$

Since $(Z, \mathcal{J}, \bar{g})$ is compact, Kähler-Einstein, with positive scalar curvature $k' = 2(2n+1)(n+1)$, the vector field X^Z is Hamiltonian, i.e.

$$X^Z = \mathcal{J} \text{grad}_{\bar{g}}(f^X)$$

where

$$f^X := -\frac{1}{2(n+1)} \text{trace}_{\bar{g}}(\mathcal{J}\bar{\nabla} X^Z)$$

is the Hamiltonian function of X^Z and $\bar{\nabla}$ denotes the Levi-Civita connection of \bar{g} . The way f^X is related to the Hamiltonian 2-form of X is explained in [3].

In order to prove that f^X satisfies the Obata's equation, we need to calculate the second covariant derivatives of \bar{X} and \tilde{A} with respect to $\bar{\nabla}$, see relation (24). There are two types of tangent vectors on Z : horizontal (i.e. which belong to the horizontal bundle determined by ∇ , seen as a connection on the twistor bundle $\pi : Z \rightarrow M$) and vertical. From Remark 8, the Obata's equation for f^X is satisfied when all arguments are vertical (the twistor lines being totally geodesic and isomorphic, as Kähler manifolds, to $(\mathbb{CP}^1, g_{\text{FS}})$). For this reason we shall not include, in Lemma 13 and Lemma 14 below, the expressions of the second covariant derivatives of \bar{X} and \tilde{A} when all arguments are vertical.

We begin with the following Lemma on the Levi-Civita connection $\bar{\nabla}$. This is by no means new, but we state it to fix notations and conventions which will be useful later on in the proof of Lemma 13 and Lemma 14. We shall denote by $\bar{\omega} = \bar{g}(\mathcal{J}\cdot, \cdot)$ the Kähler form on Z . As usual, for an admissible basis $\{J_1, J_2, J_3\}$ of Q , ω_1 , ω_2 and ω_3 will denote the corresponding Kähler forms.

Lemma 11. *For any vector fields $Y, V \in \mathcal{X}(M)$ and sections $B, C \in \Gamma(Q)$,*

$$\begin{aligned}\bar{\nabla}_{\bar{Y}}\bar{V} &= \overline{\nabla_Y V} - \frac{1}{2}(\omega_2(Y, V)J_3 - \omega_3(Y, V)J_2) \\ \bar{\nabla}_{\tilde{B}}\bar{V} &= -\frac{1}{2}\mathcal{J}(\tilde{B} \cdot V) \\ \bar{\nabla}_{\tilde{Y}}\tilde{C} &= -\frac{1}{2}\mathcal{J}(\tilde{C} \cdot Y) + \widetilde{\nabla_Y C} \\ \bar{\nabla}_{\tilde{B}}\tilde{C} &= -\langle C, J \rangle \tilde{B}.\end{aligned}$$

The above expressions are evaluated at a point $J \in Z_p$, $\{J = J_1, J_2, J_3\}$ is an admissible basis of Q , \tilde{B}, \tilde{C} are vertical vector fields on Z defined as in (23), $\tilde{B} \cdot V$ (and similarly for $\tilde{C} \cdot Y$) is an horizontal vector field on Z , which at J is the horizontal lift of $\tilde{B}_J(V_p) \in T_p M$ (here and below a tangent vertical vector $\tilde{B}_J \in T_J Z_p$ is viewed also as an endomorphism of $T_p M$ and $\tilde{B}_J(V_p)$ denotes its action on $V_p \in T_p M$).

Remark 12. In the same framework, the various Lie brackets of basic and vertical vector fields on the twistor space of a conformal-Weyl self-dual 4-manifold (i.e. a self-dual 4-manifold together with a fixed Weyl connection) were calculated in [11], Appendix A. The same formulas hold true also in the quaternionic-Kähler context, with the Weyl connection replaced by the Levi-Civita connection of the quaternionic-Kähler metric.

We now calculate the second covariant derivatives of \bar{X} and \tilde{A} as follows.

Lemma 13. *The second covariant derivative $\bar{\nabla}^2(\bar{X})$ at a point $J \in Z_p$ has the following expression: for any tangent vectors $B, C \in T_J Z_p$, $Y, U, V \in T_p M$ and admissible basis $\{J = J_1, J_2, J_3\}$ of Q ,*

$$\begin{aligned}\bar{g}(\bar{\nabla}^2(\bar{X})(\bar{Y}, \bar{U}), \bar{V}) &= \frac{1}{4}(\omega_2(Y, V)\omega_2(X, U) + \omega_3(Y, V)\omega_3(X, U)) \\ &\quad + \frac{1}{4}(\omega_2(X, V)\omega_2(Y, U) + \omega_3(X, V)\omega_3(Y, U)) \\ &\quad + \frac{1}{2}(\omega_2(X, Y)\omega_2(U, V) + \omega_3(X, Y)\omega_3(U, V))\end{aligned}$$

$$\begin{aligned}
& + g((X \wedge Y)^{S^2E}(U), V) + \frac{1}{2}\bar{\omega}(\bar{X}, \bar{Y})\bar{\omega}(\bar{U}, \bar{V}); \\
\bar{g}(\bar{\nabla}^2(\bar{X})(\bar{Y}, \bar{U}), B) &= \frac{1}{2}(g(B(\nabla_Y X), JU) + g(B(\nabla_U X), JY)) \\
\bar{g}(\bar{\nabla}^2(\bar{X})(\bar{Y}, B), \bar{U}) &= -\langle A, J \rangle g(B(Y), U) + \bar{g}(\tilde{A}, B)\bar{\omega}(\bar{Y}, \bar{U}) \\
\bar{g}(\bar{\nabla}^2(\bar{X})(B, \bar{Y}), \bar{U}) &= \bar{g}(\tilde{A}, B)\bar{\omega}(\bar{Y}, \bar{U}) - \langle A, J \rangle g(B(Y), U) \\
\bar{g}(\bar{\nabla}^2(\bar{X})(B, \bar{Y}), C) &= -\frac{1}{4}(\bar{\omega}(\bar{X}, \bar{Y})\bar{\omega}(B, C) + \bar{g}(\bar{X}, \bar{Y})\bar{g}(B, C)) \\
\bar{g}(\bar{\nabla}^2(\bar{X})(B, C), \bar{Y}) &= -\frac{1}{4}(\bar{g}(\bar{X}, \bar{Y})\bar{g}(B, C) + \bar{\omega}(\bar{X}, \bar{Y})\bar{\omega}(B, C)) \\
\bar{g}(\bar{\nabla}^2(\bar{X})(\bar{Y}, B), C) &= -\frac{1}{2}\bar{g}(B, C)\bar{g}(\bar{X}, \bar{Y}).
\end{aligned}$$

Proof. The proof uses Lemma 11 and is a straightforward calculation. The condition $W(X, \cdot) = 0$ comes into the picture by means of the Konstant formula

$$\nabla_Y(\nabla X) = R^g(Y, X) = \frac{1}{4} \left(X \wedge Y + \sum_{i=1}^3 J_i X \wedge J_i Y \right) + \frac{1}{2} \sum_{i=1}^3 \omega_i(X, Y) J_i.$$

□

In a similar way, we prove the following Lemma.

Lemma 14. *With the notations of Lemma 13, the second covariant derivative $\bar{\nabla}^2(\tilde{A})$ has the following expression at J :*

$$\begin{aligned}
\bar{g}(\bar{\nabla}^2(\tilde{A})(\bar{Y}, \bar{U}), \bar{V}) &= \frac{1}{4}(\omega_2(Y, X)\omega_3(U, V) - \omega_2(U, V)\omega_3(Y, X)) \\
&\quad + \frac{1}{4}(\omega_2(U, X)\omega_3(Y, V) - \omega_2(Y, V)\omega_3(U, X)) \\
\bar{g}(\bar{\nabla}^2(\tilde{A})(\bar{Y}, \bar{U}), B) &= -\frac{1}{2}(g(B(U), \nabla_Y X) + \langle A, J \rangle g(B(Y), JU)) \\
&\quad - \frac{1}{4}(\bar{\omega}(\bar{Y}, \bar{U})\bar{\omega}(B, \tilde{A}) + \bar{g}(\bar{Y}, \bar{U})\bar{g}(\tilde{A}, B))
\end{aligned}$$

$$\begin{aligned}
\bar{g} \left(\nabla^2(\tilde{A})(\bar{Y}, B), \bar{U} \right) &= \frac{1}{4} (g(A(U), B(Y)) - \langle A, J \rangle g(B(Y), JU)) \\
&\quad - \frac{1}{2} \langle A, J \rangle g(B(Y), JU) \\
\bar{g} \left(\nabla^2(\tilde{A})(\bar{Y}, B), C \right) &= \frac{1}{4} (\bar{g}(\bar{X}, \bar{Y}) \bar{\omega}(B, C) - \bar{\omega}(\bar{X}, \bar{Y}) \bar{g}(B, C)) \\
\bar{g} \left(\nabla^2(\tilde{A})(B, \bar{Y}), \bar{U} \right) &= -\frac{1}{2} \langle A, J \rangle g(B(Y), JU) \\
\bar{g} \left(\nabla^2(\tilde{A})(B, \bar{Y}), C \right) &= \frac{1}{4} (\bar{g}(\bar{X}, \bar{Y}) \bar{\omega}(B, C) - \bar{\omega}(\bar{X}, \bar{Y}) \bar{g}(B, C)) \\
\bar{g} \left(\bar{\nabla}^2(\tilde{A})(B, C), \bar{Y} \right) &= 0.
\end{aligned}$$

The following Lemma concludes the proof of Proposition 9.

Lemma 15. *Consider the setting of Proposition 9. Then the Hamiltonian function f^X of the natural lift X^Z of the Killing vector field X to the twistor space $(Z, \bar{g}, \mathcal{J})$ satisfies the Obata's differential equation. In particular, $(Z, \bar{g}, \mathcal{J})$ is isomorphic to $(\mathbb{C}P^{2n+1}, g_{\text{FS}})$ and (M, g) is isomorphic to $(\mathbb{H}P^n, g_{\text{can}})$.*

Proof. It is straightforward (from Lemma 13 and Lemma 14) to check that f^X satisfies the Obata's equation. Probably the most involved computation is to check the Obata's equation for f^X when the first two arguments of $\bar{\nabla}^2(df^X)$ are horizontal and the third is vertical. To do this calculation, we write (with the notations of the previous Lemmas) at J :

$$\begin{aligned}
\bar{\nabla}^2(df^X)(\bar{Y}, \bar{U}, B) &= \bar{g} \left(\bar{\nabla}^2(\tilde{X})(\bar{Y}, \bar{U}), JB \right) - 2\bar{g} \left(\bar{\nabla}^2(\tilde{A})(\bar{Y}, \bar{U}), B \right) \\
&= \frac{1}{2} \langle B, J_2 \rangle ((\nabla X)(Y, J_2 U) - (\nabla X)(U, J_2 Y)) \\
&\quad + \frac{1}{2} \langle B, J_3 \rangle ((\nabla X)(Y, J_3 U) - (\nabla X)(U, J_3 Y)) \\
&\quad + \langle A, J \rangle (\omega_2(Y, U) \langle B, J_3 \rangle - \omega_3(Y, U) \langle B, J_2 \rangle) \\
&\quad + \frac{1}{2} \left(\bar{\omega}(\bar{Y}, \bar{U}) \bar{\omega}(B, \tilde{A}) + \bar{g}(\bar{Y}, \bar{U}) \bar{g}(\tilde{A}, B) \right).
\end{aligned}$$

On the other hand, since $Y \wedge J_2 U - U \wedge J_2 Y$ and $Y \wedge J_3 U - U \wedge J_3 Y$ is J_2 (respectively, J_3) anti-invariant,

$$\begin{aligned}
(\nabla X)(Y, J_2 U) - (\nabla X)(U, J_2 Y) &= 2 (\langle A, J \rangle \omega_3(Y, U) + \langle A, J_3 \rangle \bar{\omega}(\bar{U}, \bar{Y})) \\
(\nabla X)(Y, J_3 U) - (\nabla X)(U, J_3 Y) &= 2 (\langle A, J \rangle \omega_2(U, Y) + \langle A, J_2 \rangle \bar{\omega}(\bar{Y}, \bar{U})).
\end{aligned}$$

We now easily get

$$\bar{\nabla}^2(df^X)(\bar{Y}, \bar{U}, B) = \frac{1}{4} (df^X(\mathcal{J}B) \bar{\omega}(\bar{Y}, \bar{U}) - df^X(B) \bar{g}(\bar{Y}, \bar{U})),$$

i.e. f^X satisfies the Obata's equation, when the first two arguments are horizontal and the third is vertical. To conclude the proof, it is enough to notice that Z is simply connected (see [19], Theorem 6.6) and to apply Theorem 7 to $(Z, \bar{g}, \mathcal{J})$ and the function f^X .

□

The proof of Theorem 1 is thus completed.

It is worth to relate Proposition 9 to the theory on gradient quaternionic vector fields, developed in [1] and [2]. Recall that a vector field on a quaternionic manifold is quaternionic, if its flow preserves the quaternionic bundle. On a complete quaternionic-Kähler manifold with non-zero scalar curvature, any complete quaternionic vector field is the sum of a Killing vector field and of a (complete) gradient quaternionic vector field (see Proposition 4 of [2]).

Corollary 16. *Let (M, g) be a compact, quaternionic-Kähler manifold of dimension $4n \geq 8$, with quaternionic-Weyl tensor W . Suppose there is a non-trivial quaternionic vector field X of (M, g) , such that $W(X, \cdot) = 0$. Then either (M, g) is isometric to the standard quaternionic projective space or it is Ricci-flat and its universal cover has an Euclidian factor (\mathbb{H}^k, g_0) in its De-Rham decomposition.*

Proof. Suppose first that X is Killing. We distinguish two cases: $\nu > 0$ and $\nu = 0$ (as already mentioned in the proof of Proposition 2, there are no non-trivial Killing vector fields on (M, g) when $\nu < 0$). If $\nu > 0$ then (M, g) is isometric to the standard quaternionic projective space (see Proposition 9). If $\nu = 0$ then X is parallel (see [6], Theorem 1.84) and we deduce that the universal cover of (M, g) has a flat factor in its De-Rham decomposition.

Next, suppose that X is quaternionic, but not Killing. Again we distinguish two cases: $\nu = 0$ and $\nu \neq 0$. If $\nu = 0$ then the universal cover of (M, g) is complete, simply connected and the claim has been proved in Theorem 1 of [2]. If $\nu \neq 0$ then there is a gradient (non-trivial) quaternionic vector field $\text{grad}_g(f)$ on (M, g) . The existence of such a vector field on (M, g) implies again that (M, g) is isometric to the standard quaternionic projective space (see [1], [2] and also [13]). In fact, the proof of this statement when the scalar curvature of (M, g) is positive (see [1], Theorem 1) consists in showing that the pull-back of f to the total space (S, g_S) of the 3-Sasaki bundle of (M, g) (with g normalised such that $\nu = 1$) satisfies an equation considered in Theorem A of [17] and then to deduce, using the theory developed in [17], that (S, g_S) is isometric to the Euclidian sphere of radius two. These considerations imply that (M, g) is isometric to $(\mathbb{H}P^n, g_{\text{can}})$, see [1]. Alternatively, one could have noticed that the pulled back $\tilde{f} := \pi^*f$ of f to the twistor

space $(Z, \bar{g}, \mathcal{J})$ of (M, g) is the Hamiltonian function of a Killing vector field on $(Z, \bar{g}, \mathcal{J})$ (see [3], Proposition 3.1) and check instead that \bar{f} satisfies the Obata's equation of Theorem 7. \square

For Wolf spaces, the proof of Proposition 9 can be considerably simplified, by proving the following Lemma.

Lemma 17. *Let (M, g) be a non Ricci-flat quaternionic-Kähler manifold, with quaternionic Weyl tensor W . Suppose that $W(X, \cdot) = 0$ for a non-trivial (not necessarily Killing) vector field. Then the holonomy algebra of (M, g) is the entire $\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$. In particular, if (M, g) is a Wolf space, then it is necessarily isometric to the quaternionic projective space, with its standard quaternionic-Kähler structure.*

Proof. Since (M, g) is non Ricci-flat, the holonomy algebra $\text{hol}(g)$ of (M, g) contains the $\mathfrak{sp}(1)$ -factor of $\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$. In order to prove our claim, we need to show that also $(Y \wedge U)^{S^2 E}$ belongs to $\text{hol}(g)$, for any pair of tangent vectors Y and U . Let $\{J_1, J_2, J_3\}$ be an admissible basis of Q . Since $W(X, \cdot) = 0$, also $W(J_i X, \cdot) = 0$, because W , viewed as a vector-valued 2-form, is Q -hermitian. Recall that the value of the curvature R^g on any pair of tangent vectors belongs to the holonomy algebra. Using $\mathfrak{sp}(1) \subset \text{hol}(g)$, we deduce that

$$R^g(X, V)^{S^2 E} = -\nu(X \wedge V)^{S^2 E} \in \text{hol}(g) \quad (25)$$

and

$$R^g(J_i X, V)^{S^2 E} = -\nu(J_i X \wedge V)^{S^2 E} \in \text{hol}(g) \quad (26)$$

for any tangent vector $V \in TM$. It follows that if Y or U belong to the vector space $\mathcal{V} := \text{Span}\{X, J_1 X, J_2 X, J_3 X\}$, then $(Y \wedge U)^{S^2 E}$ belongs to $\text{hol}(g)$. It remains to show that $(Y \wedge U)^{S^2 E}$ belongs to the holonomy algebra when both Y and U are orthogonal to \mathcal{V} . Take such two tangent vectors Y and U . Notice that, since both $(X \wedge Y)^{S^2 E}$ and $(X \wedge U)^{S^2 E}$ belong to $\text{hol}(g)$, also their Lie bracket, which is equal to

$$[(X \wedge Y)^{S^2 E}, (X \wedge U)^{S^2 E}] = \frac{1}{16} \sum_{i,j=1}^3 g(J_i Y, J_j U) J_i X \wedge J_j X + \frac{1}{4} g(X, X) (Y \wedge U)^{S^2 E},$$

belongs to $\text{hol}(g)$, as well as the $S^2 E$ -part of this Lie bracket. Using (25) and (26) we get our claim. \square

5 The dimension of the space of conformal-Killing 2-forms

It is known that the dimension $\text{ck}_p(M)$ of the space of conformal-Killing p -forms on a Riemannian manifold (M, g) is always finite whether M is compact or not [20]. In this Section we determine $\text{ck}_2(M)$, when (M, g) is quaternionic-Kähler and compact. We begin with the following considerations on quaternionic-Kähler manifolds with zero scalar curvature.

Remark 18. Let (M, g) be a compact quaternionic-Kähler manifold of zero scalar curvature. Being Ricci-flat, (M, g) has a finite Riemannian covering $(T^{4q} \times \bar{M}, g_0 \times g_1)$, where T^{4q} is a $4q$ -dimensional torus, with flat metric g_0 , and (\bar{M}, g_1) is compact and simply connected (this is a result of Cheeger and Gromoll, see [6], page 169). Since (M, g) is quaternionic-Kähler, (\bar{M}, g_1) is hyper-Kähler and can be decomposed into a Riemannian product

$$\bar{M} = S_1 \times \cdots \times S_l$$

where S_i are hyper-Kähler, irreducible, of dimension $4r_i$ and $\text{Hol}(S_i) = \text{Sp}(r_i)$. The Deck group G of the covering $T^q \times \bar{M} \rightarrow M$ is included in the isometry group of $(T^q \times \bar{M}, g_0 \times g_1)$ and hence is a product group $G = H \times I$ (because the metric of $T^{4q} \times \bar{M}$ is a product metric). Defining $F := T^{4q}/H$ we obtain a new Riemannian, finite covering of (M, g) , isometric to

$$N = F \times S_1 \times \cdots \times S_l, \tag{27}$$

with the following properties:

1. F is a flat manifold finitely covered by a hyper-Kähler torus;
2. the Deck group of the covering $N \rightarrow M$ is I . The isometric action of I on N is the product action and is trivial on the first factor F ;
3. if M has finite fundamental group, then $N = S_1 \times \cdots \times S_l$ (see [6], Corollary 6.67(a), page 168) and M is hyper-Kähler if and only if $M = N$, or, equivalently, if and only if M is simply connected.

Because each S_i is simply connected $b_2(N) = b_2(F) + \sum_{i=1}^l b_2(S_i)$. Of course $b_2(F)$ equals the number of parallel 2-forms on F , while on each factor S_i there are exactly three parallel 2-forms – those coming from the hyper-Kähler structure – because otherwise the holonomy would be strictly contained in $\text{Sp}(r_i)$ (see also [10], Proposition 3.15). This implies that, for a compact quaternionic-Kähler manifold with $\nu = 0$ and *finite* fundamental

group, there are no parallel 2-forms in the subbundle $(\Lambda_0^2 E \otimes S^2 H) \oplus S^2 E$. In fact, since the curvature of $\Lambda_0^2 E \otimes S^2 H$ vanishes, in the terminology of [21] we actually have $b_{\text{expt},2}(M) = 0$ in this case.

A final observation before stating our next result is that a compact $4n$ -dimensional hyper-Kähler manifold with holonomy *equal* to $\text{Sp}(n)$ is simply connected (see [6], Lemma 14.21 and also [10], Remark 4.1). In particular, a smooth finite quotient $\hat{S}_i = S_i/\Gamma$ of a hyper-Kähler manifold S_i from the decomposition (27) cannot be hyper-Kähler, unless Γ is trivial. It follows that the number of parallel 2-forms on \hat{S}_i is at most one, and is one precisely when \hat{S}_i is Kähler.

We now state the main result of this Section. We remark that our treatment when the scalar curvature is negative is not complete; however, all known (to us) *compact* examples of quaternionic-Kähler manifolds with $\nu < 0$ are locally symmetric.

Proposition 19. *Let (M, g) be a compact quaternionic-Kähler manifold of real dimension $4n \geq 8$ and reduced scalar curvature ν .*

1. *If $\nu > 0$,*

$$\text{ck}_2(M) = \begin{cases} (n+1)(2n+3) & \text{if } M \text{ is standard } \mathbb{H}P^n \\ 1 & \text{if } M \text{ is standard } G_2(\mathbb{C}^{n+2}) \\ 0 & \text{otherwise.} \end{cases}$$

2. *If $\nu = 0$,*

$$\text{ck}_2(M) = b_2(F) + 3l$$

if and only $\pi_1(M) = \pi_1(F)$ - i.e. $M = N$ in Remark 18. Otherwise, $b_2(F) \leq \text{ck}_2(M) \leq b_2(F) + 3l - 2$ and examples can be constructed to show that every possible value does occur.

3. *If $\nu < 0$ and the universal covering \tilde{M} of (M, g) is symmetric,*

$$\text{ck}_2(M) = \begin{cases} 1 & \text{if } \tilde{M} \text{ is the non-compact dual of } \text{Gr}_2(\mathbb{C}^{n+2}) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since any Killing p -form on a compact quaternionic-Kähler manifold is parallel [21], we essentially have to count parallel 2-forms unless M admits a conformal-Killing 2-form ψ which is non-Killing. By Theorem 1 this happens only when M is the standard $\mathbb{H}P^n$ in which case ψ is given as in Proposition 2

with $u = 0$ (because $b_2(\mathbb{H}P^n) = 0$) and the S^2H -component of ψ is a non-zero solution of the twistor equation. By Lemma 6.5 of [19], the composition

$$\psi \rightarrow \psi^{S^2H} \rightarrow \delta(\psi^{S^2H})$$

is an isomorphism from the space of conformal-Killing 2-forms to the space of Killing vector fields on $\mathbb{H}P^n$, so that $\text{ck}_2(\mathbb{H}P^n) = \dim(\text{Isom}(\mathbb{H}P^n))$. In all other cases $\text{ck}_2(M)$ is the dimension of the space of parallel 2-forms. We shall treat the cases $\nu > 0$, $\nu = 0$ and $\nu < 0$ separately, as follows.

Consider first the case $\nu > 0$. Recall that the space of parallel 2-forms on $M = \text{Gr}_2(\mathbb{C}^{n+2})$ is generated by the Kähler form and is one dimensional. Moreover, recall that $b_2(M) = 0$ when M is compact, with positive scalar curvature and is not isometric to the Grassmannian $\text{Gr}_2(\mathbb{C}^{n+2})$ (see [14]). This concludes the case $\nu > 0$.

The case $\nu = 0$ easily follows from Remark 18. Examples of Ricci-flat compact quaternionic-Kähler manifolds (M, g) with all possible values of $\text{ck}_2(M)$ are provided by products of finite quotients of $K3$ -surfaces.

When $\nu < 0$, certainly no solution of the twistor equation exists on (M, g) (see [9], Theorem 9); the cohomology of M is the direct sum of the $\text{sp}(1)$ -invariant and Exceptional cohomology (see [21], page 402); furthermore $b_{\text{expt},2}(M) = 0$ at least for $n \geq 3$ (see [21], Proposition 6.8). In any case, it follows from relation (8) that any parallel 2-form on (M, g) is a section of S^2E . Consider now the special case when the universal covering $\tilde{M} = G^*/K$ of (M, g) is symmetric. Parallel 2-forms on (M, g) lift to parallel 2-forms on \tilde{M} . These, in turn, are provided by 2-forms preserved by the holonomy representation of K . In particular, parallel 2-forms on \tilde{M} are in one to one correspondence with parallel 2-forms on the dual G/K of \tilde{M} . Our last claim follows from the considerations we did in the case $\nu > 0$.

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